The Paradox of the Poncelet Vanishing Point

by Pierre Beaudry

Jean Victor Poncelet

The Paradox of the Poncelet Vanishing Point—Part I

CAN YOU SOLVE THIS PARADOX?

by Pierre Beaudry

For several decades now, Lyndon LaRouche has been talking about the role of mathematical discontinuities, or singularities, as very small magnitudes which are crucial with respect to the creative process of the human mind. The point that LaRouche has been emphasizing is that these singular little creatures are not only the best means of cleaning up the cobwebs in the attic, and getting rid of unwanted axiomatics, but they have also been, since Nicholas of Cusa, the best grenades to throw in the foxholes of the Enlightenment.

So, the aim of this little pedagogical exercise is to test the reader's mind against such **enlightened pessimism** as brought about by Augustin Cauchy, by pushing to the limit certain of the infinitesimals that underlie projective geometry. The reason for this is simple: The mind can only be authoritative if it is comfortable with such paradoxes, and your sense of personal identity will be strengthened, for day-to-day organizing, proportionately to your ability to deal with such questions.

The following exercise is a partial examination of one of those singularities, the vanishing point of perspective, which is usually taken as self-evident in drawing manuals, but which Jean-Victor Poncelet had consciously used against Augustin Cauchy's stultification and denial of the creative mind. This kind of exercise will help revive, we would hope, some of the ideas that were at the center of the education program of the Ecole Polytechnique, but were destroyed by Cauchy after 1814. As Poncelet insisted, himself, in the opening statement of his class on "Geometry and Mechanics Applied to the Arts" at the Conservatory of Paris in 1826:

"Some people began to believe that mathematical truths were by necessity unintelligible to simple workers, because they are presented in abstract and difficult forms from dogmatic schoolbooks; some believed that they could not be easily understood and palpable: They were wrong. It was just that their method was at fault. There exists no mathematical principle, applicable to the works of art, that one cannot, with a little bit of study, manage to render easily intelligible to any individual with an ordinary intelligence. . . ."

Lazare Carnot

Let us start our inquiry, then, with the immediate source of Poncelet's inspiration, Lazare Carnot. In the course of investigating the mathematical infinite of the calculus, in the footsteps of Godfrey Leibniz, Carnot raised a very important paradox with respect to the metaphysical nature of infinitesimal magnitudes, which is applicable either to infinitely large magnitudes or infinitely small ones:

"There exists no discovery which has produced such a quick and wonderful revolution in the mathematical sciences, than that of infinitesimal analysis; none has given us more simple and efficient means for penetrating the knowledge of the laws of nature, by decomposing, so to speak, all bodies and magnitudes down to their constituent elements, it seems to have pointed to the internal structure and organization, but like everything that is extreme, it goes beyond our senses and our imagination. We have never been able to form but an imperfect idea of these elements, that singular type of beings, which sometimes play a real quantitative role, sometimes have to be treated like absolutely nothing, and seem, by these equivocal properties, to be holding the middle ground between a magnitude and zero, between existence and non-existence . . . What is an infinitely small quantity? It is nothing else but the difference between two magnitudes which have their limit in a third magnitude, and by magnitude, I mean, here, an actual quantity, that is neither 0 nor 1/0."

Indeed, it is generally assumed, in accepted classroom mathematics, that the sooner you neglect these infinitesimal magnitudes, these metaphors, which, in all cases, embarrass your equations, and the sooner you reduce them to zero, the better it will be, in application, for them to disappear entirely, since no sensible error of calculation can result from their practical elimination. This is the absurdity that Augustin Cauchy had emphasized, with his limit of a function, in opposition to Carnot and Poncelet. Cauchy wrote:

"As the successive numerical values of the same variable decrease indefinitely, so as to become less than any pre-assigned given number, this variable becomes what is called an infinitesimal, or an infinitely small quantity. A variable of this type has zero as a limit."

Perpetration of Fraud

This means that you have flattened the difference between linearity and nonlinearity to zero. That is a fraud. If you calculate the area under a curve with a linear means, like calculating the area of a circle by means of an inscribed polygon, and you reduce the infinitely small difference, between the two, to zero, you have cheated, you have lied, you have perpetrated a fraud.

By extension, the same applies when you choose to eliminate similar errors in summations of integrals, without incurring any significant perceptive or objective inconveniences. And the excuse you give yourself for doing this is that you always have the choice of deciding how precisely you wish your practical results to look like. Yeah, sure—if you close your eyes, nobody will see you, right? Not seeing is not an excuse for not knowing; the difference is always the result of some formalist-pragmatic outlook.

This is how the engineer has been forced to think—that is, to limit his mind to the sense perception objective world—and the earlier he is able to discard, not only these infinitely small magnitudes, but also their implied **metaphysical embarrassments**, the happier he will be. Well, sorry fellah, but this typical "happy engineer" has just fallen into the trap of cultural pessimism; he has just been brainwashed with the typical Newton/Euler/Laplace/Cauchy calculus 101, which tells you, as Donald Phau correctly pointed out to me recently, what his teacher told him in high school: "We don't know [meaning we don't care] what happens as the limit goes to zero . . . but for our purposes, it does." This is evil! Why?

The right Socratic approach should be: "What does happen as the limit appears to go to zero, and why does it happen like that; what is its purpose?" Carnot made it very clear that, from the

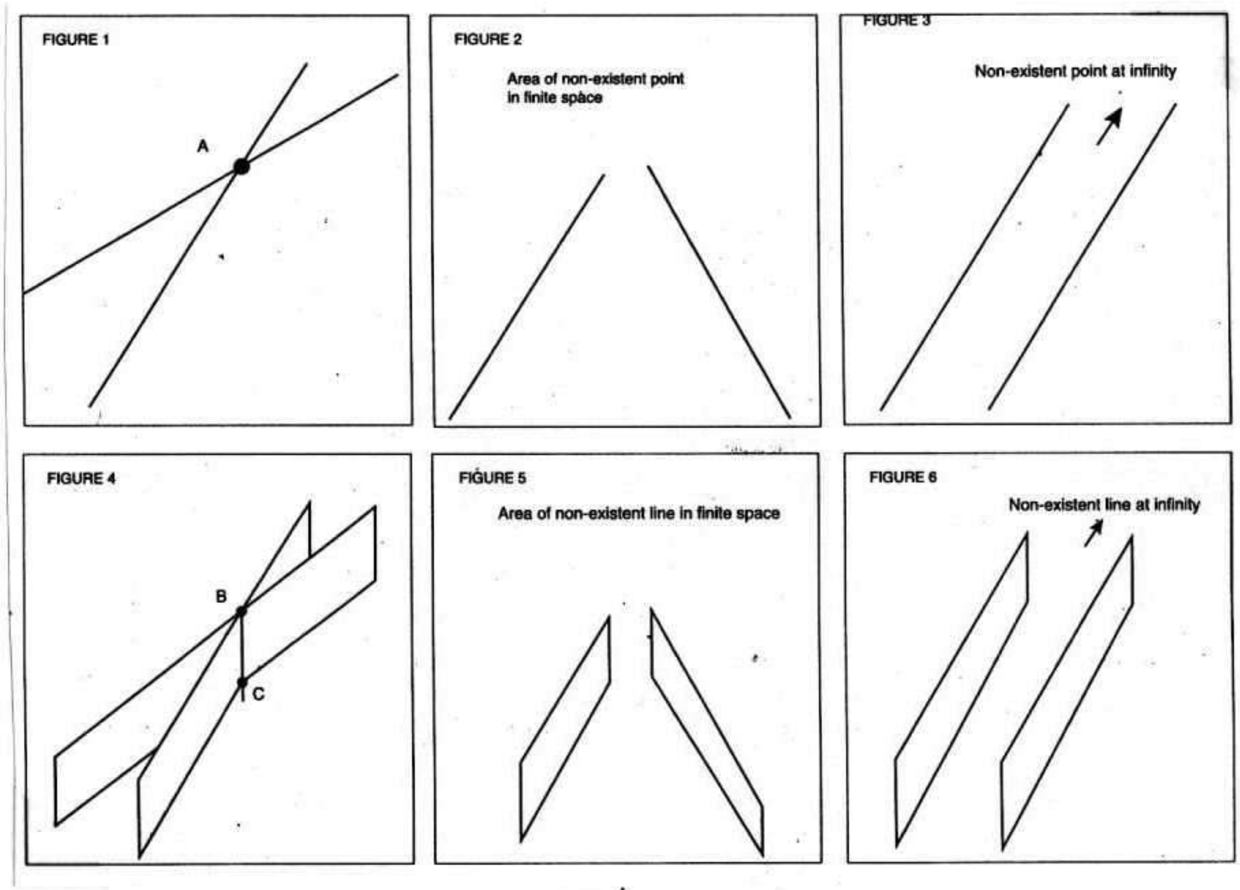
FIGURE 1
A
FIGURE 2
Area of non-existent point
in finite space
FIGURE 3
Non-existent point at infinity
FIGURE 4
B
C
FIGURE 5
Area of non-existent line in finite space
FIGURE 6
Non-existent line at infinity

standpoint of the engineer, the "theory of the infinite is nothing else but a calculus of errors compensated by imperfect equations." That was supposed to provoke some thinking on the part of the student. Indeed, the purpose of his Leibnizian calculus was not to make engineers, but to create scientists, to turn ordinary intelligence into genius. This meant that the purpose of the calculus was more than a useful approximation in calculation; it had a more profound *raison d'être* for the future survival of the human species.

Thus, the point here is not how rapidly I can eliminate the errors in my algebraic results, how I can improve the art of "calculation" of an area under a curve, how I can find a better calculus; but rather, how can I make wonderful use of the paradoxes and anomalies that I encounter in the course of investigating infinitesimals, and how can such a course bring about a discovery of principle that could be most useful for mankind? So, that will never fail to occur when you push accepted classroom mathematics to their limits. This is how we shall use the insights of Poncelet into the projective principle of perspective.

Linear and Ideal Perspective

Let us look at the question of projective geometry from the standpoint of the "principle of continuity," and see how Poncelet applies this Leibnizian principle to the so-called "point at infinity," otherwise known as the "vanishing point." Let us investigate the conditions for its very existence. Remember that Poncelet has developed the following considerations for the explicit purpose of countering the destructive impact of Laplace and Cauchy's calculus at the Ecole. So, let the warfare begin.

In first approximation, an "engineer" might say that the "vanishing point" of perspective has the useful existence of representing the limit of convergence of parallel lines, along which, three dimensional objects are represented in finite Euclidean space. Such a point is generally characterized as a geometrical object, a point which is located on a horizontal straight line that one can choose arbitrarily for the purpose of usefully replicating the three dimensional space of the "real" outside world. Let us call this approach of linear perspective, Hypothesis A. This is a hypothesis which has lasted throughout the Renaissance and the industrial revolution up until today; it is also a hypothesis which has been analytically studied at the Ecole Polytechnique, but which has been undermined by the mechanical/formalist virus of the Enlightenment.

Hypothesis B is totally different and cannot exist in the same universe as Hypothesis A. It is the hypothesis of the "scientist," or of the "savant," whose "vanishing point" is of a subjective nature and pertains to ideal perspective, which was also taught at the Ecole Polytechnique. In Hypothesis B, however, the student is urged to reject the objective world dominated by sense perception, and to reflect, for a significant concentrated period of time, on the underlying principle for the existence of certain geometric objects whose existence is purely ideal, such as objects that relate to infinitely large or infinitely small intervals; mathematical singularities which have real magnitudes, and those that have no magnitude whatsoever, and yet cannot be reduced to zero. As Poncelet insisted in his class:

"It is obviously in the phenomena of reciprocal conjunction between lines and curved surfaces that we have to seek the different characters of the non-existence of geometric magnitudes, and each of these characters is necessarily found in the mode by which that non-existence has been produced, the accident which has proceeded and accompanied that non-existence."[1]

Some of you might decide to stop reading at this point because you are thinking: "Since such objects do not exist, why the hell am I wasting my time talking about them at all? Am I some kind of a nut, or something?" Well, that is exactly the point. You are made to be stupid if you don't reduce these things to zero. The issue is precisely to develop the means of dealing with such non-existing objects, show how their non-existence has been produced, compare their non-existence, show how the condition for the non-existence of one is different than the condition for the non-existence of another, demonstrate how one is less non-existent than the other. That's the point. Only under such conditions are we going to be able to begin measuring the weight of ideas, and realize, not only the importance of their relationships, but, as well, discover how such ideal relationships are the cause of the physical universe.

Solve This Paradox

With this in mind, and if you are still willing to read on, try to solve the following paradox. Imagine two straight lines, in a plane, which intersect in a real finite point A. **(Figure 1.)** Separate slowly and continuously one of those lines from the other until you reach the position where the two lines become distinct from one another **(Figure 2)**, and then become parallel. **(Figure 3.)** The point that connected them before their separation has moved slowly outward, and has now completely disappeared, and we may consider that it is now at infinity. This same point has now become non-existent, and it now belongs to a new species of non-existent geometric objects.

Similarly, consider that, by extension of the principle of continuity, you may take two planes which intersect one another in one line BC **(Figure 4)**, and separate them until they also become distinct from one another **(Figure 5)**, and then become parallel. **(Figure 6.)** The system of those parallel planes will, by extension, find their directed points on a straight line at infinity. And furthermore, you may determine the existence of other parallel lines in other parallel planes, such that these planes will be directed, also, toward other parallel non-existent points at infinity. You may project such planes from Earth or from Mars, their distance to infinity will always remain the same. And, by doing so, you are attributing, to a system of parallel lines and planes, ideal points and lines of convergence which are purely non-existent, which lie at a distance greater that any given distance, and which are actually parallel to all other non-existent points and lines at infinity.

Now, the place for such non-existing points and lines at infinity is, so far, entirely undetermined and unlocatable, even though we know it is the same for all parallel points and lines; that is, we know they are on the same surface at infinity, no matter where I project them from. The only problem is: we don't know where that surface is. We only know it is at infinity. The question, therefore, is: How can we determine the locus of those non-existent points and lines?

(To be continued.)

Notes

1. "Applications d'Analyse et de Geometrie Qui Servent de Fondement au Traite des Proprietes Projectives des Figures" [Applications of Analysis and Geometry Which Serve as the Foundation to the Treatise of the Projective Properties of Figures], Paris, 1862, Mallet-Bachelier, Tome I, p. 345.

The Paradox of the Poncelet Vanishing Point—Part 2

CAN YOU SOLVE THIS PARADOX?

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The reason why the vanishing point in perspective does not exist is not primarily because two parallel lines can't meet. It might appear to be the case, but it is not. And besides, it could be easily shown that two parallel lines do meet, and, in that case, their converging points are very real geometric objects: the north and south poles of a sphere. So, parallelism is not the fundamental issue of perspective, it is a derivative.

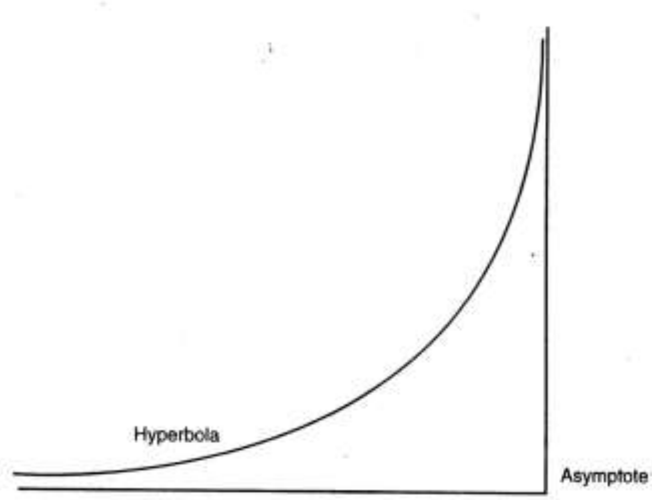
The reason why the vanishing point at infinity does not exist is because the horizon upon which it is supposed to lie does not exist. Now, think about this for a while. In order to locate a point at infinity, you must create a horizon. The horizon principle becomes the crucial *sine qua non* condition for the existence of the vanishing point. It is the horizon from which you determine the ideal intersection of parallel lines and planes that establishes, by extension, the intersection of two straight lines into a real point. It is the infinite which determines the finite! A geometric proof of this will appear in an upcoming pedagogical exercise.

In other words, unless you introduce the idea of time-reversal causality, that is, unless you create a projection into the future that will establish the determination of a horizon, an objective to reach, a purpose to achieve, an end that is hoped to be realized, a boundary condition to be satisfied, or even a function that is expected to be performed, the vanishing points or lines at infinity will have no reason to be the non-existent objects that we have described. The reality of this paradox of the non-existing point is obviously beginning to take some importance, because its validity and its resolution rest entirely on the relationship of cause and effect, and this is the kind of non-existence that the increase in relative population density of human beings depends on.

Convergence at Infinity

It is only by determining a definite infinite, or a horizon principle, that one can create the locus of a point at infinity; which is another way of saying that a definite infinite may be created, not by joining parallel lines, but by making possible that parallel lines and planes converge on a straight line at infinity. And, such a line at infinity is nothing else but an indefinite straight line whose two ends are joined at infinity. It is the One of the Many.

FIGURE 1



Indeed, the creation of a "horizon" in the art of drawing, as was established by the Renaissance geniuses like Brunelleschi, Piero della Francesca, and Leonardo da Vinci, was made as a conscious determination of the infinite, and a very powerful machine-tool design principle, a very necessary one which was not merely for the purpose that the engineer had in mind.

This idea of a "horizon" permitted to develop a first approximation of the calculus which is born out of the same type of investigation, by its creator Leibniz. In other words, perspective is actually generated by the meeting of two different species of lines, a straight line and a circle; that is, the projection of an infinite straight line (an infinite radius), meeting an infinite horizon, which is nothing else but the infinite great circle of an infinite spherical surface (Nicholas of Cusa).

Vanishing Asymptotic Point

Now, imagine another "vanishing point" at infinity, known as the vanishing asymptotic point, that is, the point toward which converge an asymptotic straight line and an hyperbola. (**Figure 1.**) Is that point the same as the "vanishing point" of perspective? If you think about this for a long enough period of time, you will discover that the answer is no. The vanishing asymptotic point is not an ideal non-existent point, but a point whose very nature is to "tend to exist," and never gets to become a point. Why? Because it vanishes at the limit of the theorem lattice.

If the asymptote tends toward the hyperbola at a distance which is absolutely infinite, it is clearly the case that this magnitude is as **infinitely distant** as the asymptote and its branch are **infinitely close** to one another; which means that, because it could always be possible to increase the distance further, by the same reasoning, it were also possible, proportionately, to subtract one more infinitely small distance between the asymptote and the hyperbola. That is, the relative addition and subtraction would come to signify that the point where the asymptote and the hyperbola would meet, had to be at a distance so great that it could not possibly exist, or be conceived, because the difference between them could always be made smaller than any given magnitude, the which could never be reduced to zero.

In other words, the vanishing asymptotic point is not a point that could acquire some existence when the asymptote and the hyperbola were to finally meet, because their function is to never meet, but to always tend toward one another at infinity. For that reason, the vanishing infinitely small space between the asymptote and the hyperbola becomes more and more evanescent, not as a geometric object that is identifiable after the difference between them has vanished, but as representing a distance that is in the process of vanishing just before the two lines meet. Thus, that point is a non-existing geometric object which can

FIGURE 2

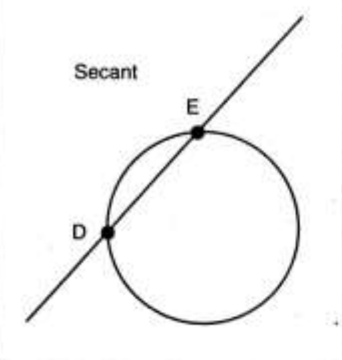
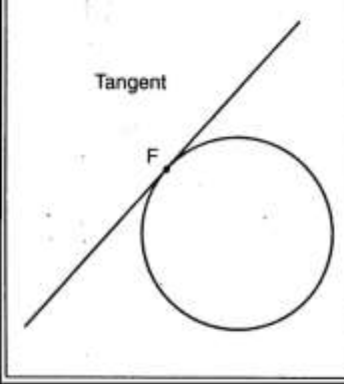


FIGURE 3



only exist as "not capable of existing," that is, an "impossible point."

So, you see, parallel and asymptotic functions do not meet the same requirements at infinity. And besides, the asymptote and the hyperbola could never come to intersect as parallel lines do. Isn't that interesting: two parallel lines never converge in the plane, yet they have a non-existing point at infinity, while the asymptote converges on the hyperbola, and yet they don't have a non-existing point at infinity.

Now, this does not mean that the contact of the hyperbola and the asymptote is impossible. It means simply that, were they to meet in conception at infinity, even though they never do in reality, the asymptote, the hyperbola, and their point of contact would either become an indefinite straight line, or implode, and **vanish altogether** into oblivion, just like the coming banking crash of 1997. So, this further reinforces the idea that the asymptotic point is a point that cannot exist, not even ideally, because it is always in the process of becoming, never comes to be, and if it came to be, at that very moment, it would immediately evaporate with the whole system, causing an axiomatic trans-infinitesimal discontinuity.

Differences in Continuity

Thus, one can see clearly that the non-existence of the vanishing point in perspective is quite different than the non-existence of the vanishing asymptotic point, and therefore the principle of continuity applies differently in both cases. In the case of the "vanishing point" of perspective, it is the real point which vanishes into a non-existent point, while in the case of the "vanishing point" of the asymptote and the hyperbola, there exists no real point, and no corresponding non-existent point, but only a non-existent point which never comes to be, or becomes transformed into an indefinite line, or comes to vanish when the system evaporates.

Imagine next, another case, a secant which intersects a circle in two real finite points D and E. (**Figure 2.**) If you rotate this secant continuously to the point where the two intersection points on the curve become one, and the distance between them has completely vanished; in order to conserve their ideal non- existence, as two distinct ideal points of conception, we may say that the interval between them (the side of the polygon) has become smaller than any other given distance, and that the secant now has an infinitely small contact with the curve, and thus becomes a tangent to it at point F (**Figure 3**), to form with it a contact of the first order, at that location. Poncelet describes this as follows:

"Since our straight line, in this new position, has become a tangent of the corresponding curve, if we continue to apply the continuous rotation away from its original position (a secant), it comes to be that the two intersection points which we have considered approaching one another in an infinitely small distance, will suddenly loose, simultaneously, their geometric existence, because the straight line will have been detached from the corresponding portion of the curve; and then, for the purpose of maintaining for them an ideal existence, as a sign in language and in conception, we will say that "these two points have simultaneously become **imaginary**, as well as the distances which separated them from real given points;" and thus is established the idea of an indefinite continuity within the common intersection of these two lines.

"The system of these two points could remain in that imaginary state during a more or less important interval, and consequently for a series of successive positions through which the mobile straight line carries them, up to the point where it becomes again a tangent, and then a secant again, and so forth in some periodical manner. However, it were possible that this straight line goes entirely to infinity; in which case all of the points of its intersection with the curve will obviously be at infinity, themselves; as a matter of fact, the line will then become tangent to the curve, have common imaginary points with it, just like secants have at finite distances, and all of this would result immediately, again, from the recognition of the principle of continuity."[1]

And, Poncelet notes that even when such points at infinity cease to be geometric objects, they are not to be confused with those that were initially real, as opposed to those which were imaginary, in the finite domain: their relationships are **extended** as singularities, as mathematical discontinuities, by the principle of continuity, and reflect different species of points that are still attached to the theorem lattice to which they belong. In fact, the points at infinity which are extended from finite geometric points Poncelet called "**real points at infinity**," while those which are still continuously connected with the imaginary ones, in the finite, he called "**imaginary points at infinity.**"

New Optimistic Axioms

Finally, reflect on the nature of the vanishing asymptotic point, and consider the shock of the implosion of the system as an actual occurrence of the collapse of the international banking system in Europe, the U.S., and Japan, simultaneously with the hyperbolic collapse of the speculative bubble of the Asian Tigers' currencies in Thailand, Malaysia, and Indonesia. What happens to the minds of the population as the boundary layer of the system breaks down, and all of this fictional capital evaporates? As Lyndon LaRouche stressed recently in a short note entitled, "On the 'Pearl Harbor' Effect, or How Paradigm Shifts Operate," a set of axiomatic beliefs, such as "I let my money do the work for me," or "Who cares about farmers? I buy my food at the supermarket," are totally changed, and are transformed into relatively new and optimistic axioms aimed at the improvement of productive labor, which then become the basis for a new belief-system, and a New, Just World Economic Order.

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Jean-Victor Poncelet

By Pierre Beaudry

Poncelet wrote in 1818, 'I have to admit that because I base all of my researches on the principle of continuity in geometry, and all of the metaphysical consequences that that entails, I am fearful that I am undermining the usually accepted ideas, and, consequently, I will not get the support that I need. . . .'

PART III
The Pedagogy of the Principle of Continuity

In the opening statement of his class on "Geometry and Mechanics Applied to the Arts," at the Conservatory of Paris, in 1826, Jean-Victor Poncelet identified the challenge of the Leibnizian **principle of continuity** in the general form of a **method of paradox** to be introduced everywhere in the schools of a republican education system.

"There exists no mathematical principle applicable to the works of the arts that one cannot, with a little bit of study, manage to render easily intelligible to any individual with an ordinary intelligence. To demonstrate this truth, I will not choose as examples the elementary principles of simple geometry, nor the least complicated mechanical combinations. I shall choose mathematical laws that have taken learned people 50 centuries of investigations to discover.

". . . I would say to the pipefitter, the plumber, the boilermaker, the lathe operator: When you make a diagonal cut across a pipe, a roll, or a funnel, you create an oval [or elliptical] cut; and you gardener, you trace the same ellipse with a rope and posts. Now, suppose that your ellipse is more than 200 million *fathomes* long; replace one of the posts by an eternally gleaming ball, a Sun which is 1,348,460 times larger than the Earth; and finally, make the Earth itself roll along an elliptical pathway at a speed of 23,000 [fathoms†] per hour. Then you shall have an idea of the immense force with which the Almighty moves one of the smallest globes of one of the smallest worlds—worlds which include as many suns as you can imagine there are countable stars in the universe as a whole. Then, trace around that post, the center of the Sun, as many ellipses as there are planets, and incline their planes to a greater or lesser degree, and make them according to the length and width that I give to you in numbers, and there you shall trace the pathways of the planets; and finally, each planet is the sun of its satellites, and the focus of their elliptical orbits.

"That is how we shall make easily understood to workers, the magnitude of our solar system and of the masses that compose it, with such simple, beautiful, shall I say divine, ordering of the eternal movements that underlie these phenomena. This idea they will acquire in a few minutes, but I say again, it took centuries for disciplined people, respected for their works of art and science, to elevate themselves to the same level of knowledge."

So, as you can see, the importance of this paradox of the vanishing point is not to be found so much in the nature of the point itself; this point is just the footprint of something else that is going on, and which is properly located in the domain of the underlying ordering Leibnizian principle of continuity, which depends on a more general Leibnizian principle which states that: "As data is ordered, so the unknowns are ordered also." All of these points, the **real point at infinity** of perspective, the **vanishing asymptotic point at infinity** of the hyperbola, the **imaginary points at infinity** of the secant, and so forth, belong to different species which are generated by the same ordering principle of continuity, by virtue of which Leibniz considered that "the excluded extreme [at infinity] can always be treated as included." It is the mental nature of such a locus of transformations which is our concern here in this series of pedagogical exercises.

Notes

1. "Applications d'Analyse et de Geometrie Qui Servent de Fondement au Traite des Proprietes Projectives des Figures" [Applications of Analysis and Geometry Which Serve as the Foundation to the Treatise of the Projective Properties of Figures], Paris, 1862, Mallet-Bachelier, Tome I, p. 350.

This is the principle of the Monge brigades; it is precisely this education policy that LaPlace and Cauchy and other mathematicians hated so much about the use of the Leibnizian method at the original Ecole Polytechnique. They utterly despised the fact that ordinary workers should be so privileged, and that ordinary intelligence could understand, so quickly, such "difficult things." What these "mathemagicians" secretly brooded on, but would never dare say in public, is: "How can a worker with an ordinary intelligence grasp these questions of **infinitesimals** and **indivisibles**, while we, the elite of the intelligentsia, cannot even understand what this is all about?" This is the truth that Professor M.O. Terquem was deathly afraid to admit in his correspondence with Poncelet.[1]

Augustin-Louis Cauchy

Poncelet's Enemies

The 'mathemagicians,' LaPlace and Cauchy hated the use of the Leibnizian method at the Ecole Polytechnique. They utterly despised the fact that ordinary intelligence could understand, so quickly, such 'difficult' things, while we, the 'elite' of the intelligentsia, cannot even understand what this is all about."

Pierre-Simon Marquis de LaPlace

In the field of geometry, as in political intelligence, it is always very instructive to discover how your adversary thinks, and why he thinks like that. In his reply to Poncelet, Terquem wrote the following telling statement, in which he reveals that the analytic mind cannot grasp the idea of the **vanishing point** that Poncelet is projecting to infinity. Wrote Terquem:

"In fact, you show us, by using the example of the intersections of two lines, . . . that many intersection points disappear as a result of their separation; real magnitudes become imaginary, and you say that these imaginaries correspond precisely to algebraic imaginaries . . . to what algebraic impossibility must correspond this geometric impossibility we have no notion, and we are very eager to have your clarification on this sudden passage from being to non-being. We have no knowledge of anything similar in analysis."

And, to show how far a formalist-pragmatic mind will go to fill in the gap left by the lack of ideas, Terquem ends his reply by asking Poncelet for some "useful applications"—to which Poncelet replied that he might have had better results if he had asked of him to "give an application of the fable where the mountain gives birth to a mouse."

Poncelet really had a war and a half on his hands, because even people who were friendly to him, would not accept his ideas. They simply could not understand why he would go to so much trouble to create non-existent geometric objects, and encumber their minds with such trifles. Typical "sincere" critics, like Terquem, were convinced that Poncelet was honest in his endeavor, but totally "over-imaginative, and impractical." In point of fact, the whole matter was over their heads, because they refused to make the effort to clean the cobwebs out of their attics, and dared not risk the unknown. They would not admit that the limitation, and the flaw, lay in themselves.

However, this is no mere trifle. This pedagogy is absolutely needed in the education system for two essential reasons: First, this method was a means of "discovery and of invention." As Poncelet said, "We do not intend to teach you a method and a process for each art, but instead, what is the principle common to all of the arts . . . with the purpose of making inventors of you, inventing new machines and new processes." Second, this is a method of solving the Ontological Paradox of Plato, the Parmenides Paradox of the One and the Many. Since the ordinary intelligence of man is naturally attracted by the infinite, it is fitting that it measure itself against ever greater generalization, and integration of data; because the greater the access to simplification and truth on the part of the power of reason, the greater its increasing proportionality with respect to the Great Architect, the more man **becomes** in the Image of God. And this is what the generalization of the principle of continuity will help achieve, as a means of testing how far the mind can go in its quest for the Absolute. So, in that sense, this principle is used as a litmus test, as Leibniz put it himself, to discover who is thinking, and who is not. Leibniz defined his principle as follows:

"This principle (of continuity) has its origin in the **infinite** and is absolutely necessary in geometry, but it is effective in physics as well, because the sovereign wisdom, the source of all things, acts as a perfect Geometer, observing a harmony to which nothing can be added. This is why the princi-

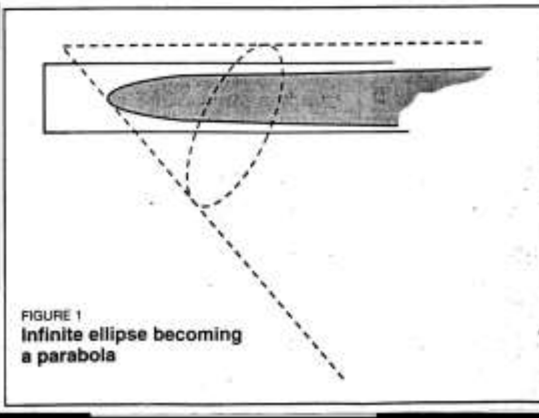


FIGURE 1
Infinite ellipse becoming a parabola

ple serves me as a test or criterion by which to reveal the error of an ill-conceived opinion at once and from the outside, even before a penetrating internal examination is begun. It can be formulated as follows. **When the difference between two instances in a given series, or that which is presupposed, can be diminished until it becomes smaller than any given quantity whatever, the corresponding difference in what is sought or in their results must of necessity also be diminished, or become less than any given quantity whatever.** Or to put it more commonly [colloquially], **when two instances or data approach each other continuously, so that one at last passes into the other, it is necessary for their consequences or results (or the unknown) to do so also.** This depends on a more general principle : that, **as the data are ordered, so the unknown are ordered also.** ***(Datis ordinatis etiam quaesita sunt ordinata.)*** But examples are needed in order to understand this. We know that a given ellipse approaches a parabola as closely as we choose, so that the difference between ellipse and parabola becomes less than any given difference, when the second focus of the ellipse is far enough away from the first focus, for then the radii from that distant focus differ from parallel lines by an amount as small as we choose. And, as a result, all the geometric theorems which are proved for the ellipse can be applied to the parabola. . . ."[2]

This is precisely the issue that Cauchy had rejected, when he rebuffed Poncelet's request for approval at the Academy of Sciences. Cauchy wrote: "Strictly speaking, this principle (of continuity) is nothing else but a strong induction, by means of which one extends theories that are first established in accordance with certain restrictions, to other cases in which these restrictions no longer exist. Being applied to curves of the second degree [conic sections—ed.], this principle has led the author to exact results. However, we think that it could not otherwise be accepted generally and be systematically applied to all sorts of questions in Geometry, and even in Analysis." This is precisely the point: The principle of continuity must have the most universal application possible.

Let us apply this litmus test to our own minds, and see if we can solve the enigma that Leibniz has posed for us. What is the difficulty? What is the paradox that has to be resolved? The question is: **How can an ellipse become a parabola?** The immediate obvious answer seems to be: There is no such thing as an ellipse that can be a parabola, under any circumstance. An ellipse is an ellipse, and cannot be something else. That is the logic of non-contradiction that Terquem, or Cauchy, would use to avoid the dilemma. "A thing is what it is, according to the principle of identity, and cannot be something else, at the same time, by virtue of the principle of non-contradiction." The problem with this Aristotelian logic is that the **case at infinity** has not been taken into consideration. Let us think about this for a little bit.

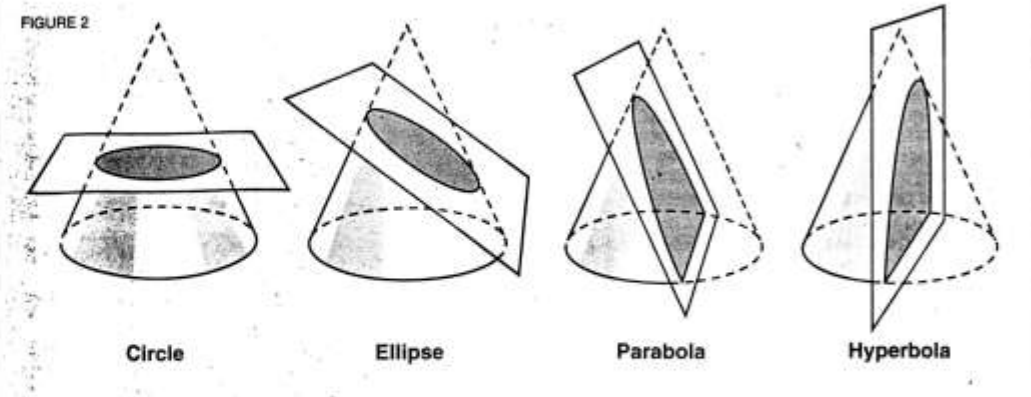
Let us examine more closely the particular elliptical case that Leibniz is considering. What does he mean when he says, "When the second focus of the ellipse is removed far enough from the first focus"? How far is "far enough"? That is the key. Yes, he is implying that the second focus of the ellipse is at infinity. He is implying that the only way to pass, in a continuous way, without a jump, from the ellipse to the parabola, is for the ellipse to lose its second focus; or—which comes to the same thing—for the ellipse to become infinite, thus transforming itself into a new conic figure, the parabola, which then becomes the limiting case of that infinite ellipse.

Although by moving a plane progressively through a cone you will generate, without any jumps or difficulties, a continuous geometric transformation from an ellipse to a parabola, the necessity for the infinite to be introduced in the process, by virtue of the principle of continuity, causes a major discontinuity in the mind of the beholder. Disbelief and doubts set in, and linear and formal logic break down. **"An ellipse at infinity is a parabola—what an outrageous idea! You are an extremist!"** To make this point more tangible, consider the following figure of a plane cutting a cone (Figure 1).

Indeed, when the plane cutting the cone is infinitely close—as close as you choose—to becoming parallel to one of the sides of the cone, the conic section is an infinitely long ellipse which is becoming a parabola. At the point that the plane has become parallel, then the second focus of the ellipse is at infinity, and has become non-existent; at that precise moment, the ellipse has also become non-existent, because it has been transformed into a parabola! Thus, this non-existent ellipse at infinity is nothing but a parabola.

This is why Leibniz asserts that the rules for the ellipse must also apply to the parabola, because of the continuous transition of a plane, which cuts the cone through an uninterrupted rotating and sliding progression, across the cone, without making any jumps between any of the conics: that is, when the plane moves up and down, while it is perpendicular to the axis (and parallel to the base) of the cone, **the conic is a circle;** when the plane is tilted and cuts the cone from side to side, **the conic is an ellipse;** when the plane travels from side to side across the cone, while it is parallel to one side of the cone, **the conic is a parabola;** and when the plane cutting the cone moves from side to side across the cone, while it remains parallel to the axis of the cone, **the conic is a hyperbola** (Figure 2).

FIGURE 2



We shall see, in the following pedagogical exercise, that Poncelet applied the very same principle to the passing of a circle into an ellipse, and vice versa. He defined his principle of continuity precisely in accordance with Leibniz:

"The axiom that we are examining and considering is, from a certain point of view, nothing else but the **principle of permanence**, or of **indefinite continuity of the mathematical laws of magnitudes that vary by imperceptible succession**, a continuity which often subsists in a purely abstract and ideal manner for certain conditions of the same system....

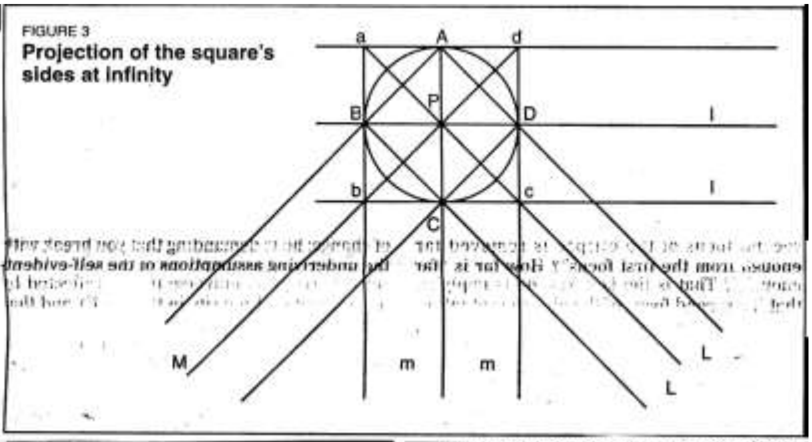
"The principle of continuity, considered from the standpoint of geometry, means that, **if you conceive of a given figure whose situation is in the process of changing, according to some progressive and continuous movement of its parts, without violating in any way whatever the dependence and the relation that was established initially between those parts, the relations or the** metrical properties which related to the figure in the initial situation remain applicable, in their general form, to all of the derivative figures, without any other change except the simple denominations of **plus** and **minus** which can intervene between them within their relationships. As for the purely graphic or descriptive relationships concerning the given primitive figure, they maintain their application to all of the derivative figures without any modification except for those which occurred in the respective situations of lines.

"It is from the simple observation that, in geometry, (non-existent) beings of reason can only be created from the willful extension of the law of continuity, even for the cases where the conjunction of lines is physically impossible, that I have come to establish the proper and distinct characters that belong to them, respectively. I have derived from the same examination different metaphysical notions which I consider new and important; as, for instance, the following: **In space, all of the points at infinity must be considered, from the standpoint of continuity, as being distributed on a plane at infinity whose situation is indeterminate.**"[3]

Now, examine these concepts and apply them to a situation where your mind is bringing the system of a given species to a formal limit which is not zero, but which is an **infinitesimal discontinuity**, an unbridgeable gap which you must leap over in order to go beyond into the system of a new species, as if into a new manifold; the passing of an ellipse into a parabola will be perfectly continuous, geometrically speaking, and will occur without a jump, but, what happens in the mind is a crucial **discontinuity**. This is the paradox that Poncelet is addressing: **The principle of continuity in geometry leads to the discontinuity of a jump function in the mind.**

Here, the discontinuity Poncelet is forcing your mind to go through is to make the leap from the analytical-algebraic and formal system of linear thinking to a synthetic geometry of change; he is demanding that you break with *the underlying assumptions of the self-evident* sense-perception universe that is reflected by linear algebra (linearity in the small), and that you challenge your mind with non-existent geometrical objects, as well as the paradoxes that they represent, at the limit of the system, i.e., **at infinity**.

FIGURE 3
Projection of the square's sides at infinity



PART IV
How the Principle of Continuity Leads to a Disconinuity Function

Let us take an example of how the principle of continuity applies to the ellipse and the circle. As Leibniz admitted for the case of an ellipse passing over into a parabola, the same law shall apply when an ellipse passes into a circle. That is, by some continuous motion of a plane intersecting a cone, we can project an ellipse to come as close to a circle as we wish, such that, when the distance between the two foci of the ellipse becomes less than any infinitesimal amount you choose, the two foci become one, and the ellipse passes into that circle which has that double "focus" as its center; then, the concurrent tangents of the ellipse become parallel tangents to the circle, and the intersection point of the concurrent tangents becomes a non-existent point at infinity. Let us demonstrate this as follows:

Draw a circle with an inscribed quadrilateral *ABCD* and a circumscribed quadrilateral *abcd* (Figure 3). Extend the sides and the diagonals of the two quadrilaterals, three by three, such that they form four sets of parallel lines directed toward the non-existent points at infinity, *M, L, m, l*. How can you locate those non-existent points *M,L,m,l*, at infinity, by using the principle of continuity?

As we have seen earlier, such non-existent points at infinity can only be located by creating a **horizon**, an infinite straight line, on which all of those points lie. For this reason, Poncelet has established that "following the principle of continuity, **all of the points at infinity, projected from a plane, can be ideally considered as being distributed on a single straight line, itself located on that plane at infinity.**"[4] Assuming that Figure 3 is a plane figure, how can its **horizon** be a straight line, when the points *M,L,m,l* are going in four different directions? By making use of the **horizon principle**, find the projective property which will transform those four directions of parallel lines in such a way that they will converge harmonically onto one finite straight line.

Passing from an Ellipse to a Circle at Infinity

We have just made the claim that it is possible to pass from an ellipse to a circle without any leap, according to the Leibniz-Poncelet **principle of continuity**; but this has the effect of causing in the mind, a **discontinuity**, a paradox whereby a system of parallel lines at infinity can be transformed into a system of concurrent lines in finite space. How do we solve that paradox?

Project, from a point *M*, a cone which is tangent to a conic section, such as an ellipse

FIGURE 4
Contracted infinite points projected onto an ellipse

(Figure 4). As Poncelet put it, "For example, draw from point *M* two tangents *ME* and *MG* to the conic curve, and trace from the same point two arbitrary secants *MBA*, *MCD*; then join, by new straight lines, the intersection points *A,B,C,D*. Those lines will intersect in points *P* and *L*, whose cord of contact is the polar *EG* of point *M*."[5] This completes the construction of the inscribed quadrilateral. If you now join points *L* and *M* by a straight line, you have constructed the **horizon**, and you have located the extension *ML* as a segment of the sought-for straight line at infinity. The infinite line that was impossible to draw in Figure 3 is now very real and finite in Figure 4.

Next, to construct the circumscribing quadrilateral *abcd* (Figure 5), all you need to do is to extend the diagonals *AC* and *BD* of the inscribed quadrilateral of Figure 4, until these lines meet line *ML*. Then project, from these new points of intersection *m* and *l*, two pairs of tangents, *mB,mD*, and *lA,lC* touching the conic curve. (This is quite beautiful, and the reader must construct his own model of this, in order to really savor the full joy of the discovery.) The intersections of these four tangent lines will form the circumscribing quadrilateral *abcd*.

By comparing Figures 3 and 5, one can see that there exists a reciprocity between a system of parallel lines and a system of concurrent lines. Poncelet wrote:

"Any given figure which comprises a system of straight or curved lines which have a common intersection point, can always be considered as the projection of another figure, of the same type or order, in which the intersection point has passed to infinity, and whose corresponding lines have become parallel."

And, by virtue of the principle of continuity, the reciprocal:

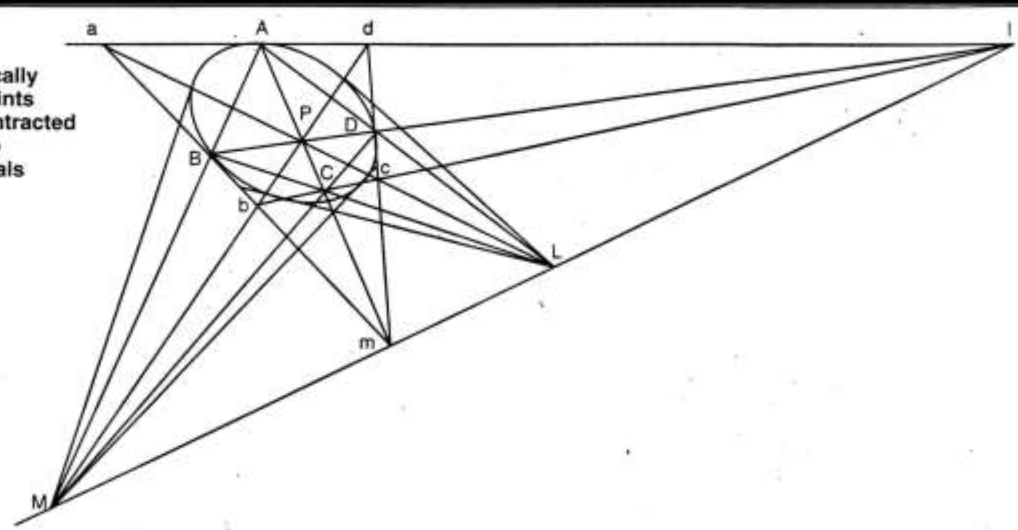
"A plane figure which comprises a system of straight and curved lines, parallel or concurrent at infinity, has in general on a given plane, another projected figure of the same order in which the corresponding lines are concurrent at one common point at a finite distance, as a projection of the first system."[6]

Thus, you have resolved, by applying the principle of continuity, the central paradox of the projective properties of figures of the same order, such as the circle and the ellipse.[7] But then, another paradox arises from this: In passing from the circular system to the elliptical system, the four sets of parallel lines have been transformed from an indeterminate infinite range into a contracted infinite harmonic range between the four conic projections *M,L,m,l*. How can the non-existent point at infinity become the source of proportionality in the finite domain? This is what we shall see next.

To be continued.

FIGURE 5

Set of four harmonically conjugated finite points projected from a contracted infinite line onto two elliptical quadrilaterals



Notes

1. M.O. Terquem was one of several academic judges to whom Poncelet wrote, in November of 1818, for publication approval of his manuscripts. He was Poncelet's former mathematics teacher at the Museum of Artillery in Metz.

2. Gottfried Leibniz, "Philosophical Papers and Letters," Kluwer Academic Publishers, Vol. 2, Boston, 1956, p. 351.

3. J.-V. Poncelet, "Applications of Analysis and Geometry, Which Serve as the Principal Foundation for the Treatise on the Projective Properties of Figures," Gauthier-Villars, Paris, 1864. pp. 533-534.

4. Poncelet, op. cit. p. 49.

5. J.-V. Poncelet, "Treatise of Projective Properties of Figures," Gauthier-Villars, Vol. I, Paris, 1866, p. 97.

6. J.-V. Poncelet, idem., Vol. I, Section I, Chap. III, Art. 101-102 p. 51.

7. J.-V. Poncelet, "Treatise on the Projective Properties of Figures," Gauthier-Villars, Paris, 1865, Introduction, p. XXII. Poncelet describes these properties in the following way :"The special objective that I am proposing in writing this book is to enlarge the resources of simple geometry by generalizing their conceptions and language, which are usually quite restricted, but most of all to develop the general means of discovering and demonstrating, in an easy fashion, that class of properties which pertains to figures, when we consider them from a purely abstract standpoint which is independent of any absolute and determined magnitude. I have made the claim that such properties exist for a given figure as well as for all of its projections or perspectives; and I have differentiated them from all others with the generic term of **projective properties**, which, in a nutshell, relates to their true nature.

"Since those projective properties must, unquestionably, belong to the most general properties that can be known, they deserve, by this sole qualification, the total attention of geometers; in fact, we know that those properties of extension are the more fruitful in curious consequences, or useful in practice, when their statements are contained in the most general, simplest, and easiest to grasp. We also know that, because of the indeterminacy that characterizes them, such properties subsume implicitly, and as direct corollaries, all the other particular properties of figures."

* The English word is Pleiad, and the meaning is a group of seven illustrious persons—after the seven sisters of Greek mythology who were changed into stars (hence the constellation the Pleiades). The use of the word in French is especially common, since the 1556 adoption by the poet Ronsard of the word *Pléiade* to describe himself and six colleagues.

** All emphasis within quotes, in original text.

† The archaic French measurement *fathome* is not to be confused with the English marine measurement fathom. In terms more accessible to the English-speaking reader, the Earth, in its annual motion around the Sun, moves at roughly 6,600 miles per hour.

Poncelet's Co-Thinkers and Allies

Poncelet identified the challenge of the Leibnizian **principle of continuity** in the general form of a **method of paradox** to be introduced in the schools of a republican education system: 'There exists no mathematical principle applicable to the works of the arts that one cannot, with a little bit of study, manage to render easily intelligible to any individual with an ordinary intelligence.'

Gottfried Leibniz — Library of Congress

Gaspard Monge — AIP Niels Bohr Library

Alexander von Humboldt — Library of Congress

Elements of Biography of Jean-Victor Poncelet

An Initital Attempt to Salvage Poncelet's Work

The folloiwng partial biography is based on notes published by Poncelet in his "Treatise on the Projective Properties of Figures."

As Poncelet himself admitted, he was apprehensive about his entire project of developing a theory of projective geometry based on the Leibnizian principle of continuity, because he was conscious of undermining the whole edifice of the Academy of Sciences and their accepted classroom mathematics. As he wrote to one of his academic judges, M. O. Terquem, professor of mathematics at the Museum of Artillery in Metz, on Nov. 23, 1818: "I have to admit that because I wanted to base all of my researches on the admission of the principle of continuity in geometry, and all of the metaphysical consequences that that entails, I am fearful that by giving them birth, I am undermining the usually accepted ideas, and, consequently, I will not get the support that I need from the enlightened men whom I have chosen as my judges" (p. 538). In point of fact, Terquem, who did not dislike Poncelet, as Cauchy did, did not understand three-quarters of what he wrote, and finally recommended that he publish only a quarter of what he had written.

Although Poncelet never stopped fighting for the recognition of the importance of his ideas, he would nonetheless admit that his principle of continuity had virtually no chance of being accepted by that generation of accepted classroom mathematicians in France.

"This discussion brings me naturally to show how the doctrine of the **correlations of figures** (Carnot), which extends so considerably the domain of synthetic geometry, is itself founded on the mere tacit admission of the principle of extension that we have just defined and examined. I end this topic by making the observation that the principle of continuity has not been accepted in geometry with all of the generality that it deserves, but only in certain favorable circumstances where it did not counter generally accepted ideas, since, otherwise, it would have been charged with metaphysical considerations of imaginaries, the which have always been kept out of the narrow sanctuary of geometry: which leads me to observe that, if we wish to be as rigorous and logical as the Ancients, we have to banish altogether that principle from geometry, that axiom, or otherwise, admit it with all of the generality that it implies, without worrying about the singular and paradoxical consequences that may result from its application—just as was done in algebraic analysis, where these difficulties still exist but without hampering its march nor its progress" (in Jean-Victor Poncelet, "Applications of Analysis and Geometry, Which Serve as The Principal Foundation of the Treatise on Projective Properties of Figures," Mallet-Bachelier, Paris, 1862, p. 532).

As early as the first publication in 1822, Poncelet knew that he would have to export his ideas outside of France, if he wanted them to survive. That very admission demonstrates that the soul of France had been destroyed by the Cauchy calculus of limitations, the calculus of castration, as Lyndon LaRouche called it.

The origin of the influence of the Ecole Polytechnique outside of France is difficult to date, and the political situation of the French Revolution makes it uncertain as to how rapidly scientific and philosophical ideas were being exchanged across the continent during the period. This is exemplified by the case of Lazare Carnot taking up residence at Magdeburg, Germany, in 1816. It was during that year that the Prussian professor Wilhelm Koerte wrote the first Carnot biography, published in Leipzig, in 1820. This is probably the first published account of the Ecole Polytechnique's ideas in Germany.

Even with the Bourbon restoration, the political expulsion of Monge from the Ecole in 1815, and the perverse Entente Bestiale [the Entente Cordiale] of the Treaty of Vienna, a crucial conduiting of these ideas was successful through the good offices of the physicist, Alexander von Humboldt. It was Humboldt who intervened to recruit a *Pléiade** of French scientists and artists, among them Jean-Victor Poncelet, Jean-Baptiste Biot, Augustin Jean Fresnel, and Francois Arago, who were all investigating geometry, astronomy, optics, and magnetism, and whom he made members of the Order of Merit of Prussia—in other words, he made them honorary Germans.

As early as 1808, Humboldt visited Polytechnique student Poncelet at the infirmary of the Ecole, where he had taken ill. Those were the years of intense collaboration between Humboldt and Louis-Joseph Gay-Lussac, doing experimental work on optics and magnetism in the laboratories of the Ecole; Fresnell and Etienne-Louis Malus confirm Huygens' hypothesis on the wave front propagation of light by discovering polarized light; Arago, Malus, Biot, Fresnell, and Humboldt all collaborate in studies of refraction of light through the Earth's atmosphere.

From that moment on, Humboldt would keep in close contact with Poncelet, until the 1830s, meeting often at Poncelet's place, or at Arago's home, or even at General Baugrand's residence, where they discussed, among other things, the works of Jacobi, then exiled in Koenigsberg, the works of Abel of Christiana, the ideas of Fortunato Padula of Naples, and those of Jacob Steiner, who had become *Privatdocent* (Assistant Professor) in Berlin.

There is no doubt that such collaboration also directly intersected the work that Gauss and Weber were doing in Germany on magnetism, since Humboldt was in direct contact with Ampere in Paris, at the Ecole, and had met Poncelet in Metz, where he had travelled for the purpose of making experiments on the Earth's magnetic field. Poncelet admitted himself that, "In these intimate spiritual and scientific exchanges, I have found in Alexander von Humboldt a devoted friend and protector. . . ."

A most fruitful collaboration was also initiated when Humboldt introduced Poncelet's works to Dr. Crelle, the founder of the Journal of Pure and Applied Mathematics, in which, for the first time, the writings of Poncelet on the "Theory of the Projective Properties of Figures," were introduced to a German audience. In fact, Dr. Crelle praised

Poncelet's work in the very first volume of his publication. Poncelet aknowledges:

"The publication of the Crelle Journal, in 1826, was a true blessing for me, because of the unbelievable discussions that my work had provoked in France.

"Indeed, in spite of the apprehension that my critics might have inspired in him, in spite of the injustices that I suffered from such luminaries as Gergonne and Cauchy, and in spite of the spirit of rivalry that he might provoke in his own country by publishing my works, the good and loyal Dr. Crelle dared to announce with praise, in the very first installment of his publication (T. 1, January 1826, p. 96), my 'Treatise on the Projective Properties of Figures,' in a remarkable bibliographical article, which demonstrated his courage and showed his sense of equity" (Poncelet, "TPPF").

For reasons that might have been caused by some unadmitted rivalry, Jacob Steiner, who was a collaborator of Crelle, and who also published several pieces in the same issue of the Journal that year, had chosen not to acknowledged the importance of Poncelet's ideas with respect to his own work. Obviously very hurt by this, Poncelet noted that he did not understand why Steiner could have claimed that "he had borrowed nothing from French authors before 1826" (Crelle, T. 1, p. 162). Nonetheless, Poncelet regarded Steiner's work as "a beautiful example of good geometry of the Monge school."

It is not excluded that Jacob Steiner might have had access to Poncelet's projective geometry work prior to 1826, and might have studied the Poncelet theorems of projective geometry that students were circulating from Paris to Berlin at the time, especially those on the relationship of geometric figures and the infinite. Steiner's General Theory of Tangency and Intersection of Circles and Spheres, and his work on harmonic ranges, attest to this kind of Poncelet approach, specifically in the theorems where Steiner projects infinitely large and infinitely small circles. Steiner uses the same principle of continuity that Poncelet had borrowed from Leibniz, to articulate a **continuous mapping** of an infinite circle onto a finite one; thus, providing a geometrical approach to the study of discontinuities and their behavior in passing from the finite to the infinite, and vice versa.

Highlights of Poncelet's Public Activities

Jean-Victor Poncelet (1788-1867) is born in Metz, France, and graduates from the Ecole Polytechnique in November of 1810. In 1812, as an officer in Napoleon's Army, he joins the great army of Vitebsk, and is taken prisoner in Russia, where he will be imprisoned at Saratoff until 1814, when he returns to France. From 1815 to 1825, Poncelet has the opportunity to put together the geometry work he had written in prison.

First publication of "Treatise on Projective Properties of Figures" occurs in 1822. In 1823 and 1824, the Inspector Generals of Artillery and Engineering Corps, MM. Vale, Beaugrand, and Arago, who are examiners at the School of Application in Metz, propose the creation of a class on the Science of Machines, which Poncelet will start teaching in 1825.

Poncelet's commitment to teach ordinary workers is exemplified by the fact that he began a class of Applied Mechanics in Industrial Sciences, at the City Hall of Metz. These were free public lectures given to artists and workers concerning the geometry of curves applied to the arts as well as to industry. The polemical nature of these classes is made explicit in the public flyer that advertises the sale of the teacher's notes at a local bookstore:

"The properties and the tracing of curves other than the circle, are based on theories of advanced mathematics, which can only be found in voluminous and erudite volumes, quite inaccessible to the general public. The author will expose them in a manner that will be easily understood by artists and workers who have taken the class of 'Applied Geometry for Industry,' and which will serve as its complement. Thanks to this book, it will no longer be required to master analytical geometry, in order to know how to trace the **ellipse, the hyperbola, the parabola, the catenary, the figure eight curves, the spirals, the evolutes and involutes, the cycloid, epicycloid, convex curves**, etc., whose uses are as widespread as that of the circle. As we did in the first class, this second class will also provide you with many applications with respect to Arts, Industry, and to the understanding of the laws of natural phenomena."

During that period of 1823-25, Poncelet is sent to visit and report on the advancement of technological application in the manufactures of France, Belgium, Germany, and England.

- **1825-1830**—Poncelet teaches at the Artillery School, and at the City Hall of Metz. He invents the famous curved paddle-wheels, which are in standard use today, in hydraulic turbines.
- **1830-1834**—He is a member of the City Council of Metz, and secretary to the General Council of the Moselle Department. He is nominated Member of the Academy of Sciences of the Institut, and in charge of the scientific evaluations of fortifications.
- **1838-1848**—Poncelet becomes professor of geometry at the Faculty of Sciences in Paris, and creates the class of Experimental and Physical Mechanics at the Sorbonne.
- **1848-1851**—He becomes a Member of the Constituent Assembly, and is nominated professor at the School of Administration, and the College de France. He is Commanding Officer of the Ecole Polytechnique, and of the National Guards of the Seine Department. In 1848, Poncelet unsuccessfully attempts to reform the curriculum of the Ecole Polytechnique.
- **1851-1858**—Poncelet is nominated president of the tools and machinery sections of the Universal Expositions of London and Paris. He is in charge of the historical report on machine-tools for the first Universal Exposition. He is sent on exploration trips for the study of silk, linen, and hemp industries across France.
- **1862-1864**—Publication of "Applications of Analysis and Geometry, Which Serve as The Principal Foundation to the Treatise on the Projective Properties of Figures."

We have seen, in the previous pedagogical exercises on the projective geometry of Jean-Victor Poncelet, that the Leibnizian principle of continuity enabled us to establish the non-existence of certain singularities at infinity, and then to carry them back as ideal geometric objects into the finite domain. We have also established, in the last segment of Part IV of this series, that a new paradox emerged in the process of passing from a circular system to an elliptical system—a paradox whereby it was possible to transform an indeterminate infinite range into a contracted infinite harmonic range between four conic projections. In this concluding part of the series, we shall develop the principle of how such harmonic proportionality is actually generated from infinity.

PART V
Harmonic Proportionality Generated from the Infinite

Consider the beautiful proportionality of the four harmonically ordered conic points **M, m, L, l** (Fig. 1) projected from a contracted infinite straight line onto a given ellipse. The ordering principle of the quadrilaterals **ABCD** and **abcd** of the inscribed and circumscribed quadrilaterals is such that all of the intersections of the 16 tangents and secants of the two figures, projected from finite points, are actually generated from the infinite. That is, the harmonic ranges formed by the intersections of all of the secants and tangents are entirely determined from the non-existent points at infinity.[1]

First, establish that the infinite non-existent straight line at infinity has become a contracted infinite which generated the following harmonic range in the finite:

Ml:Ll :: Mm:mL

Next, establish that this primary harmonic range will generate the harmonic ranges of the four conical projections, directed from points **M, m, L, l**. Now, here is the new paradox to be solved: It is because the harmonic range **Ml:Ll :: Mm:mL** represents a **contracted infinite** that the harmonic ranges of the four cones are formed by their mutual intersections. That is, the four harmonic ranges

1) dM:bM :: dP:Pb
2) aL:cL :: aP:Pc
3) Am:Cm :: AP:PC
4) Bl:Dl :: BP:PD

Note that when you generate the ordering from infinity, it is the ratio of the infinite segments which determines the ratio of the finite segments, and not the other way around. For example, it is the infinite ratio of the two infinite segments **dM** and **bM** (in the parallel lines of the circular figure) which determines the finite ratio of **dP** and **Pb** of the same figure. (See Fig. 2, "Projection of the square's sides at infinity.") In other words, it is because **dM:bM** = 1 that **dP:Pb** = 1; and this means that the two infinite segments correspond to unity, in such a way that the ratio of their **indivisibility** is equal to 1.

Now, this implies another fascinating paradox, which has accompanied us from the very beginning, which has been with us throughout all of these exercises, which some readers may have noticed. I bring to your attention the crucial paradox of the line whose two ends are, respectively, finite and infinite, at the same time; the very case of infinite parallel lines.

One of the implications of this curious paradox is that you cannot add or subtract a finite portion to, or from, an infinite line. Or, to put it differently, you can add or subtract a finite portion to and from the finite part of an infinite line, but without changing the total magnitude of the line. You cannot modify the totality of an infinite line which has a beginning that is infinite and an end that is finite! Now, think about this for a moment: The infinite distance **dM** (Fig. 2) is longer than the infinite distance **bM** by as much as the finite distance **db**, and yet the two infinite distances are the same distance. How can you add to something without changing it? When you add a finite segment to a finite line, you are making that line grow; yet, when you add a finite portion to an infinite line, you are not making the infinite line grow. Why not?

This is another curious consequence of the principle of continuity with respect to **indivisibles** that Poncelet identifies as follows:

"If two infinite distances or magnitudes differ only by a given finite quantity, their proportion will be unity; such that they can be considered strictly equal to one another".[2]

This means that even though the infinite segments **dM** and **bM** differ by a finite amount **db**, their infinite ratio **dM:bM** will be equal to unity. Furthermore, and in its ontological capacity of being the primary ratio, that infinite ratio will also represent the ordering principle of every other ratio; that is, the infinite ratio is the ratio from which all of the other ratios originate. Consequently, this also means that the finite cannot affect the infinite—in a sense, in the way that the One cannot be affected by the Many.

Now those are just some of the consequences of the principle of continuity, so resolving this paradox will require a little bit of attention and scrutiny here.

Think of the way in which Leibniz identifies

FIGURE 1

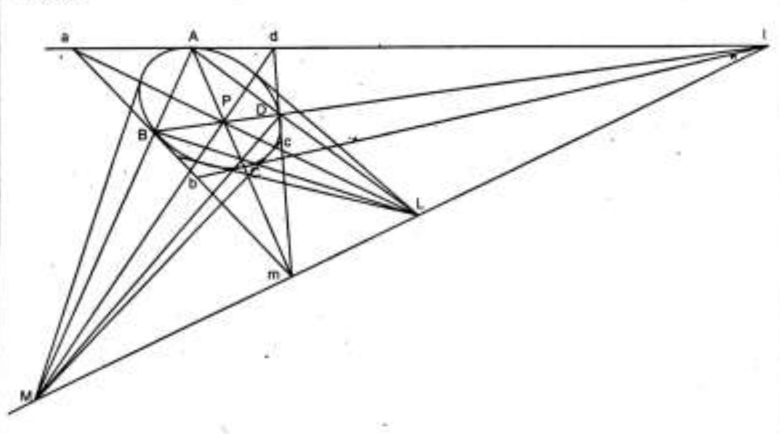
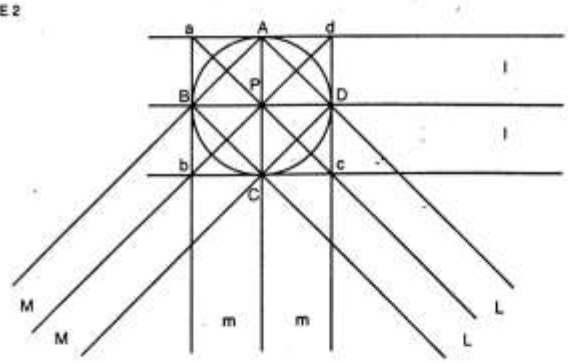


FIGURE 2



what he termed an "indivisible." For example, a beginning cannot be divided, because it is not extended. So, all non-existent beings at infinity are such non-extended beings, and therefore are not affected when you add something to them, or when you subtract something from them. Follow closely Leibniz's reasoning:

"**There are indivisibles or unextended beings**, for otherwise we could conceive neither the beginning nor the end of motion or body. The proof of this is as follows. There is a beginning and an end to any given space, body, motion, and time. Let that whose beginning is sought be represented by line **ab**, whose middle point is **c**, and let the middle point of **ac** be **d**, that of **ad** be **e**, and so on. Let the beginning be sought at the left end of **a**. I say that **ac** is not the beginning, because **cd** can be taken from it without destroying the beginning; nor is it **ad**, because **ed** can be taken away, and so forth. So, nothing is a beginning from which something on the right can be removed. But that from which nothing extended can be removed cannot be extended. . . . There is no point whose part is 0, or whose parts lack distance; whose magnitude is inconsiderable; incapable of being designated, less than that which can be expressed by a ratio not infinite to another sensible magnitude; less than any which can be given."[3]

FIGURE 3

This is the only optimistic use that can ever be made of this otherwise evil Xeno's Paradox. Yes, there is a **sufficient reason** for the beginning not to be divisible, and that reason is that an "indivisible" is of a different species (a contradictory species), even though it is attached to a divisible segment. The same **sufficient reason** is applicable to parallel lines that go to infinity: The parts of them which are non-existent, and therefore not extended (which is the longest part of them, since the finite part is only the tip at the end), are of a different species, and will not be divisible either, nor will they admit additions.

In other words, the infinite portion of an infinite line which has a finite end cannot be added to, because the infinite cannot be increased or divided by any finite portion. This means that, by making the finite portion longer or lesser, you are not making the infinite portion lesser or longer, proportionately. No proportion of the finite can affect the **proportionality** of the infinite. However, the finite portion, which is connected to that same infinite line, can be as long as you wish, and it can be divided, or added to, at will.

Now, how can this be? How can it be that we have one continuous line, one end of which can be added to, and the other end of which cannot?

Since the ratio we are looking at is infinite, even when it is attached to a finite portion of a line, it remains something to which nothing extended can be added. But, you might ask, why is there a real line attached to it? How can this be, since that to which nothing extended can be added, cannot itself be extended? How therefore, can two infinite parallel lines even exist? Now, you are beginning to see that between them there exists, paradoxically, an incommensurable gap. You might wish to say that a line like this should not even exist, should be like zero. Because how can a line be continuous and discontinuous at the same time?

That is precisely the point: Parallel lines projected to infinity do exist precisely in that paradoxical form, as non-existent lines with an existent end.

So, it is precisely for that reason that our infinite first ratio of equality must be an "indivisible" or an "unextended being," in the sense stated above by Leibniz—and therefore, it must be considered as **expressible** or **designateable**, or made **intelligible**, as the last, or first of all possible inequalities, just as the state of rest is the first of all motions, an infinitely small motion.

Consequently, according to the law of continuity, Leibniz asserts that "it is permissible to consider rest as an infinitely small movement, (that is to say as equivalent to a species of its contradictory,) and coincidence as an infinitely small distance, and equality as the last of inequalities, etc."[4]

PART VI
The Infinite as The Source of Harmony In the Infinite

In bringing to a close this series of pedagogical exercises in projective geometry, it would be useful to recall here the objective Poncelet set for himself in his classes, following the model of the *Monge Brigades*, and to emphasize the importance of replicating, around the world today, such a policy of education, because what the elite of the intelligentsia will not grasp, the ordinary intelligence of workers will.

Hence we quote here from Poncelet's introductory speech to his Class on Industrial Mechanics, for workers and artisans.[5]

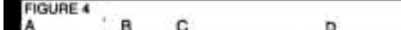
"There is no mechanics without geometry . . . and the most important thing to grasp and to master, is the sentiment [higher emotional sense] of proportionality, that is, the theory of proportions or of the equality of relationships that you are considering. . . . We will not teach you methods and procedures specific to each art, but rather teach you what is universally common to all of the arts, and what serves as principle to all of them. . . . In this regard, I am counting on the good will, on the zeal of all of you, and particularly of those among you who have a better mastery of our mathematical method, to encourage and help those who have more difficulties. Indeed, it is by attempting to make someone else understand the principles of science that we can achieve a better understanding ourselves.

"But our classes will at least have served to warn you against chimeras, against the fantasies of a deranged imagination which, by missing the sure principles of science, behaves as if it had created wonders that sane reason will disavow. . . . These classes should also serve you as warnings against the charlatanism of diplomas, against the phony guarantees of so-called specializations, of apparent inventions which have not been sanctioned by experience, and through the transparency of which you can immediately see the fallacy; these classes will help you make the distinction, after sufficient examination, between what is true and what is false, between what is possible and what is not possible, between what is good and really useful, and what is not; they will convince you, finally, that the arts become more perfect gradually; that the discoveries which excite the admiration of our century, have come about by other similar discoveries, and have often cost a whole lifetime of work, and even the lives of several successive generations, in order to arrive at the point of perfectibility that you can witness today.

"The more knowledge you acquire, the more you will be guarded against the seductions of vanity . . . because you need to know a lot in order to recognize that there is still much more to know; and that is why a really knowledgeable man, a man of great talent, is essentially a modest man."

It is with that sense of Socratic modesty that we must, as does Poncelet in his classes, challenge our minds with one more paradox in this series—that is, the one by which we are able to grasp that it is through the singularity at infinity that we discover what causes and determines the conditions of existence in the finite domain. Once we have gained the advantage of that, it becomes easy to demonstrate to others where the true residence of ordinary intelligence lies, and how the harmonic range in the finite domain is but a pale reflection of that great harmony that resides in the mind of the Creator.

FIGURE 4



Poncelet develops a beautiful way to locate infinity, simply by means of the projective properties of points that are located arbitrarily on a straight line in the plane. Simply start by choosing any three points **A, B, C** on a straight line such that the second point **B** lies closer to the third point **C**, than it is to point **A**. You can easily find, on that same line, a fourth point **D** which will determine a harmonic range between **A, B, C, and D.** The reader should construct the following projection himself with a straight edge only.[6]

FIGURE 5

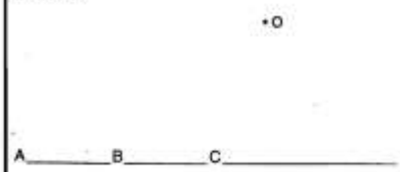
Given three arbitrary points **A, B, C** on a straight line, find the fourth harmonic point **D**.

First, project from any point **O** three rays **OA, OB, OC** in such a way that ray **OA** bounces back, anywhere through the other two rays **OB** and **OC**. Do the same with ray **OC**, and bounce it back through **OB** and **OA**, but crossing precisely through **B'**. Next, extend a straight line from **A'** through **C'**, until it reaches the extension of line **ABC** at point **D.** You have now discovered, by this simple, and beautiful, construction, the locus of the fourth harmonic point **D** of a harmonic range. Note that **D** is the only one of the four points which was not chosen arbitrarily, and it is from its discovery that we are able to generate a definite harmonic range between all four points, such that:

AD:CD :: AB:BC

Second, project on another line, again, three points **A, B, C** in such a way that point **B** is moved slowly toward **A**. What happens when point **B** reaches the midpoint between **A** and **C**?

FIGURE 6



By bouncing rays **OA** and **OC** through **B'**, as we did in the previous construction, we can no longer find point **D** anywhere on line **ABC**. Why? What has happened to point **D**? If you think for a moment about what we have done in the first pedagogical exercise of this series, you will discover that we have proven, by construction only, and without any algebraic mumbo-jumbo, that when you slowly **separate from one another** two intersecting lines **ABC** and **A'B'C'**, the point where they intersect at **D** moves outward, and reaches infinity when the two lines become parallel.

FIGURE 7

Thirdly, when you project arbitrarily three points **A, B, C** on a straight line in such a way that point **B** becomes closer to point **A** than it is to point **C**, the fourth harmonic point **D** reappears—but this time, on the opposite side of the infinite—and forms **once again a harmonic range** such that:

CD:AD : : CB:BA

What happened here? Simply by our moving point **B** by an infinitesimal increment of action from one side of the midpoint between **A** and **C** to the other, we have created a situation in which point **D** has passed, at a quasi-isochronic speed, from one side of the infinite to the other, without our noticing it. This is amazing! You have caused the moving of the non-existent point **D** from infinity to infinity by simply tilting line **A'C'** by an increment smaller than any given magnitude.

That's pretty good, because now you know, seeing with your mind's eye, that if you were to fold point **C** onto point **A**, in order to find the midpoint **B** (see Fig. 6), you would have, by way of circular action, caused the infinite plane to be folded on itself—to be divided into two halves—and you would have caused point **D** to go from one side of the infinite circle to the other! You have done **all** of that with virtually no action at all, just by bending (barely bending) the parallel plane with your mind.

Lastly, it is relevant to show that the circular **horizon** at infinity, which we have just perceived with our mind's eye, and which was made intelligible with this balancing act of the parallel line **A'C'**, is the actual source of determination of this marvelous harmony in the universe. Indeed, if we closely observe Fig. 6, where point **D** has vanished to infinity, it is clear that **A[D]** = infinity, and **C[D]** = infinity as well. Thus:

A[D] :C[D] = infinity (∞) = 1

Thus, it is because infinity is unity that the finite segments [**AB** and [**BC**] are equal, as we have established before; or it is because the infinite unity is the first of all harmonic inequalities, that the harmonic ordering is established within a range of four finite points. Therefore:

A[D] :C[D] : : Infinity (∞) : : AC : BC

This is how you are able to create non-existent beings at infinity, on the right or on the left of the infinite plane, and are able to make them intelligible; also, you are able to make them causal when, from the infinite, you generate a finite harmonic range out of the divisibility by two. Only through this axiomatic underlying ordering and division of the perspective lattice, can we derive parallel lines, as well as all of the harmonics of the finite Euclidean domain.

In conclusion, the important thing to emphasize, here in these exercises, is that **the locus of these differentiations is the locus where axiomatic changes occur in the mind.** This is the locus of **hypothesis** where ideas are formed, Platonic Ideas, like paradoxes and metaphors, where certain ideas can be transformed into others, within the same lattice, and without any leap, continuously—for instance, like the passing at infinity of the ellipse into the parabola, and the parabola into the hyperbola. Furthermore, this is also the locus of the **higher hypothesis** which subsumes entire series of discontinuous hypothesis, the Riemann-LaRouche **continuing principle of discontinuities**, which implies the constancy of paradoxes generated by a jump function where entire domains of ideas paradoxically change, at a higher level, according to the same Leibnizian principle of continuity; where the linear exclusion of the **extreme** cases at infinity is always included as non-linear singularities.

So this shows that the non-existences of singularities, discontinuities at infinity, are not all the same, whether we deal with them from within the same theorem lattice, or from outside a given lattice: Some are continuously attached to their real geometric space-time referent, while others go beyond the discontinuity of the domain from which they emerge, and form a higher domain, which is the space-less and time-less isochronic continuity domain of the higher hypothesis. This higher-hypothesis form of isochronic constancy is the ordering principle from which Leibniz, Carnot, and Poncelet locate enthusiasm (*agape*) as going "... further than wisdom, without exceeding its region. . . ."[7]

All emphasis throughout the article is emphasis in the original.

Notes

1. In reproducing Figure 5 of the previous part of this series (the Oct. 20th New Federalist, Vol. X, No. 41), we must apologize to the reader for the mistaken location of point **C**. The harmonic ordering is possible only when all four points **ABCD** actually touch the ellipse.

2. J.-V. Poncelet, "Treatise of Projective Properties of Figures" (Gauthier-Villars, Paris, 1864), Vol. I, p. 15.

3. Gottfried W. Leibniz, "Philosophical Papers and Letters" (Kluwer, Boston, 1989), Vol. 2, pp. 139-140.

4. G.W. Leibniz, "*Acta Eruditorum*," Letter to M. Varignon, December 1701.

5. Introductory speech by Poncelet before his Class on Industrial Mechanics, given in Metz, Nov. 5, 1827.

6. See Pierre Beaudry, "The Metaphor of Perspective," Fidelio, Summer 1995. There are many other proportions to be found in this elementary construction, but this primary one will suffice for our purpose here. For further study of constructive synthetic geometry, see also the most important follower of Poncelet in the German school, Jacob Steiner, "Geometrical Constructions with a Ruler, Given a Fixed Circle with its Center" Scripta Mathematica Studies, No. 4, 1950.

7. From Carnot's poem on "Enthusiasm."

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